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Roughness effect on the frictional force in boundary lubrication

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The influence of surface roughness on the frictional force between two walls separated by a thin liquid film is investigated. It is shown that the presence of roughness increases essentially the frictional force in confining systems and can lead to the time dependence of the friction. Both effects are related to the morphology of rough interfaces. A new dependence of the frictional force on the thickness of the liquid film is found.

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I. INTRODUCTION

Many attempts have been made to investigate friction, adhesion, and lubrication at the solid-liquid interface [1]. An increasing number of applications require an understanding of lubricants in confined geometries when liquid-film thicknesses become comparable to molecular dimensions. Advances in our understanding of the phenomenon of friction are facilitated by the recent development of new tools that allow the study of contacts at microscopic scales. These new tools are the surface-force apparatus (SFA) [2–4] and the atomic-force microscope (AFM) [5,6]. Also the quartz-crystal oscillators have been used to determine the frictional forces between a surface and an adsorbed film of one or more monolayers [7]. Different theoretical approaches [1,8–13] have been proposed to explain the microscopical features of the friction in confined geometries. Analytic models and molecular-dynamics computer simulations have provided much insight into the phenomenon of friction at the atomic scale. But so far the fundamental microscopic understanding of the phenomenon remains limited.

A common feature of all theoretical approaches is the consideration of atomically flat surfaces and the neglection of surface roughness. But real surfaces including atomically flat are always characterized by a certain roughness whose degree depends on the actual material, the method of surface treatment, and the presence of adsorbed particles on the surface. The nonuniform microscopic structure of the liquid layers strongly bounded with the surface can introduce an additional effective roughness. It was shown experimentally [14] that surface roughness can drastically affect the friction in confined geometry.

In this paper we concentrate on the question of the influence of surface roughness on the frictional force between two walls separated by a thin liquid film. The liquid flow generated by moving one of the walls is calculated by the perturbation method of Rayleigh [15] and Fano [16]. A similar approach was used recently for the description of the effect of roughness on the frequency of a quartz-crystal resonator in contact with a liquid [17].

We consider here the effect both of a randomly rough surface and a periodical corrugation. It is shown that the presence of roughness essentially increases the frictional force in the confining system and can lead to the time dependence of the friction. Both effects are related to the morphology of rough interfaces. The new dependence of the frictional force on the thickness of the liquid film is found.

II. THE MODEL

To model the experimental system we consider two rough walls separated by a thin liquid film (see Fig. 1). A liquid flow was generated by moving the bottom wall at constant velocity V_0 in the direction along the y axis. The rough interfaces between the walls and the liquid are described by the equations $z=\xi_1(x,y)$ and $z=d+\xi_2(x,y)$ giving the local heights of the walls with respect to reference planes. The reference planes $z=0$ and $z=d$ are chosen so that the spatial averages of the surface profile functions $\xi_1(x,y)$ and $\xi_2(x,y)$ vanish. Here d is the average thickness of the liquid film.

The fluid velocity $\mathbf{V}(\mathbf{r},t)$ is the solution of the linearized Navier-Stokes equation

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{r},t)=\frac{1}{\rho}\nabla P(\mathbf{r},t)+\frac{\eta}{\rho}\nabla^2\mathbf{V}(\mathbf{r},t)\ , \tag{1}$$

where $P(\mathbf{r},t)$, η, and ρ are the pressure, the viscosity, and

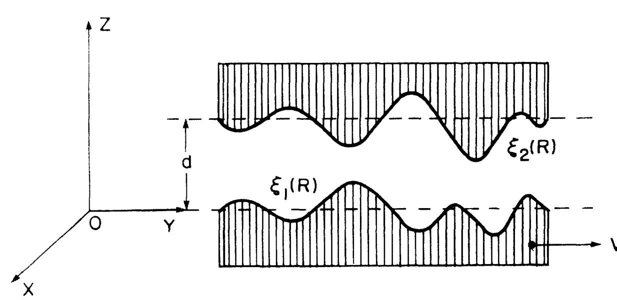

FIG. 1. A schematic illustration of the geometry used in our model.

*To whom correspondence should be addressed. Electronic address: urbakh@ccsg.tau.ac.il the density of the fluid. The velocity must also satisfy the incompressibility condition

$$\nabla \mathbf{V}(\mathbf{r},t)=\mathbf{0} \; . \tag{2}$$

We will use the stick boundary conditions for the fluid velocities at the movable $z=\xi_1(x,y-Vt)$ and fixed $z=d+\xi_2(x,y)$ interfaces.

$$\begin{aligned} &\mathbf{V}(x,y,z=\xi_1(x,y-V_0t))=\mathbf{V}_0 \; , \\ &\mathbf{V}(x,y,z=d+\xi_2(x,y))=\mathbf{0} \; . \end{aligned} \tag{3}$$

This is one of the fundamental assumptions in fluid mechanics [20]. While experiments at macroscopic scales are consistent with the stick boundary condition, recent measurements, which probe molecular scales, indicate that the boundary conditions may be different [7,21]. Molecular-dynamics simulations [10,11] showed that the degree of slip at the interface is related to the liquid structure induced by the solid wall. At large interactions between liquid molecules and the substrate the first one or two liquid layers became locked to the wall. We will focus on this case only.

We will solve the problem by the perturbation theory [15–17] which is valid for slightly rough surfaces, $|\nabla\xi_{1,2}(x,y| \ll 1$. For such surfaces the characteristic size of roughness in the z direction (the root mean square height), h, is less than the tangential one (the correlation length or the period), l. The height h denotes the root meal square departure of the surface from flatness, and the correlation length l is a measure of the average distance between consecutive peaks and valleys on the rough surface.

In order to solve Eqs. (1)–(3) it is convenient to Fourier transform the profile functions, the velocity, and the pressure from the tangential coordinates $\mathbf{R}=(x,y)$ and the time t to the corresponding wave vectors $\mathbf{K}=(K_x,K_y)$ and the frequency ω according to equations

$$\begin{aligned} &\xi_{1,2}(\mathbf{K})=\int d\mathbf{R}\,\xi_{1,2}(\mathbf{R})\exp(-i\mathbf{K}\cdot\mathbf{R}) \; , \\ &\mathbf{V}(\mathbf{K},z,\omega)=\int d\mathbf{R}\int dt\,\mathbf{V}(\mathbf{r},t)\exp[-i(\mathbf{K}\cdot\mathbf{R}+\omega t)] \; . \end{aligned} \tag{4}$$

then Eqs. (1) and (2) may be rewritten in the form

$$\begin{aligned} i\omega\mathbf{V}(\mathbf{K},z,\omega)=&-\frac{1}{\rho}\left[i\mathbf{K}+\mathbf{n}_z\frac{\partial}{\partial z}\right]P(\mathbf{K},z,\omega) \\ &+\frac{\eta}{\rho}\left[\frac{\partial^2}{\partial z^2}-\mathbf{K}^2\right]\mathbf{V}(\mathbf{K},z,\omega) \; , \end{aligned} \tag{5}$$

$$\left[i\mathbf{K}+\mathbf{n}_z\frac{\partial}{\partial z}\right]\mathbf{V}(\mathbf{K},z,\omega)=0 \; . \tag{6}$$

Here $\mathbf{n}_z$ is the unit vector in the z direction. It follows from Eqs. (5) and (6) that the pressure $P(\mathbf{K},z,\omega)$ obeys Laplace equation

$$\left[\frac{\partial^2}{\partial z^2}-\mathbf{K}^2\right]P(\mathbf{K},z,\omega)=0 \; . \tag{7}$$

The solutions of Eqs. (3) and (5)–(7) have the form [17]

$$P(\mathbf{K},z,\omega)=P_1(\mathbf{K},\omega)\exp(-Kz)+P_2(\mathbf{K},\omega)\exp(Kz) \; , \tag{8}$$

$$\begin{aligned} V_\alpha(\mathbf{K},z,\omega)=&(2\pi)^3V_0(1-z/d)\delta_{\alpha,y}\delta(\mathbf{K})\delta(\omega)+A_{\alpha,1}(\mathbf{K},\omega)\exp(-q_Kz)+A_{\alpha,2}(\mathbf{K},\omega)\exp(q_Kz) \\ &-\frac{1}{\rho\omega}K_\alpha[P_1(\mathbf{K},\omega)\exp(-Kz)+P_2(\mathbf{K},\omega)\exp(Kz)], \quad \alpha=x,y \; , \end{aligned} \tag{9}$$

$$\begin{aligned} V_z(\mathbf{K},z,\omega)=&\frac{i}{q_K}\sum_\alpha K_\alpha[A_{\alpha,1}(\mathbf{K},\omega)\exp(-q_Kz)-A_{\alpha,2}(\mathbf{K},\omega)\exp(q_Kz)] \\ &-\frac{i}{\rho\omega}K[P_1(\mathbf{K},\omega)\exp(-Kz)-P_2(\mathbf{K},\omega)\exp(Kz)] \; . \end{aligned} \tag{10}$$

Here

$$K=|\mathbf{K}|, \; q_K=(i\omega\rho/\eta+K^2)^{1/2}$$

and $V_\alpha(\mathbf{K},z,\omega)$ and $V_z(\mathbf{K},z,\omega)$ are the projections of the vector of the velocity $V(\mathbf{K},z,\omega)$ on the axes $\alpha=x$, y, and z correspondingly. The first term in the right-hand side of Eq. (9) is the solution of the hydrodynamical problem for two flat interfaces.

The coefficients $A_{\alpha,i}(\mathbf{K},\omega)$ and $P_i(\mathbf{K},\omega)$, for $i=1,2$, are obtained from the boundary conditions (3). The method of the calculation of the coefficients is described in the Appendix.

After the determination of the fluid velocity $\mathbf{V}(\mathbf{r},\omega)$ the frictional force per unit area, F, can be found from the energy balance in the system. The rate of the change of kinetic energy of the liquid, E_{kin}, should be equal to the sum of the frictional work in unit time and the rate of energy dissipation, Q, in the liquid

$$\frac{d}{dt}E_{\text{kin}}=FV_0S+Q \; , \tag{11}$$

where S is the area of the bottom wall.

It should be noted that, in the problem under consideration, the change of the kinetic energy is much smaller than the dissipation,

$$\frac{dE_{\text{kin}}}{dt}\Big/Q\sim\rho V_0d^2/\eta l \ll 1 \; ,$$

and it will not be taken into account below. The rate of energy dissipation can be written in the form

$$Q=-\frac{\eta}{2}\int d\mathbf{R}\int_{\xi_1(\mathbf{R}-\mathbf{V}_0t)}^{d+\xi_2(\mathbf{R})}dz\sum_{i,j}\left[\frac{\partial V_i}{\partial r_j}+\frac{\partial V_j}{\partial r_i}\right]^2 \; . \tag{12}$$

In order to determine the first non-negligible correction to the friction force due to the influence of roughness we have to find the coefficients $A_{\alpha,i}(\mathbf{K},\omega)$ and $P_i(\mathbf{K},\omega)$ up to the second order with respect to the root mean square height of the roughness, h. It follows from Eqs. (11) and (12) that it suffices to find the second-order corrections only for the y-component of the velocity, $V_y^{(2)}$, at zero value of the two-dimensional wave vector $\mathbf{K}$. To the first and the second orders we have

$$V_j^{(1)}(\mathbf{K},z,t)=\frac{V_0}{d}[\xi_1(\mathbf{K})\exp(iK_yV_0t)f_j(Kd,z/d,\phi)+\xi_2(\mathbf{K})f_j(Kd,1-z/d,\phi)],\quad j=x,y,z\ , \tag{13}$$

$$V_y^{(2)}(K=0,z,t)=\frac{V_0}{d^2}\int\frac{d\mathbf{K}'}{(2\pi)^2}\{\xi_1(\mathbf{K}')\xi_2(-\mathbf{K}')\exp(iK_y'V_0t)(2z/d-1)r_1(K'd,\phi)$$
$$+[|\xi_1(\mathbf{K}')|^2(1-z/d)-|\xi_2(\mathbf{K}')|^2z/d]r_2(K'd,\phi)\}\ , \tag{14}$$

where ϕ is the angle between the wave vector $\mathbf{K}$ and the axes x.

It should be mentioned that the main contribution to the final result is given by the values of K, which are of the order of the inverse correlation length (or the period) of the roughness, $1/l$. In the case of slightly rough surfaces the correlation length is approximately hundreds of angstroms or larger [18,19]. In the experiments with a surface-force apparatus[2–4] the separation between the walls, d, is of the order of tenths of angstrom. Hence the condition $d/l \ll 1$ is usually satisfied. Under this condition it suffices to calculate the functions $f_j(Kd,z/d,\phi)$ and $r_{1,2}(Kd,\phi)$ at small values of the parameter Kd. Up to the second order with respect to the parameter Kd we obtain the following expressions for the functions

$$\begin{aligned}
f_x(Kd,z/d,\phi)&=\tfrac{3}{2}(z/d)(z/d-1)\sin(2\phi)+\frac{(Kd)^2}{60}(z/d)[7+3(z/d)-25(z/d)^2+15(z/d)^3]\sin(2\phi)\ ,\\
f_y(Kd,z/d,\phi)&=z/d+3(z/d)(z/d-1)\sin^2(\phi)\\
&\quad+\frac{(Kd)^2}{60}(z/d)(1-z/d)\{-3-15(z/d)^2+[15(z/d)^2-10(z/d)-7]\cos(2\phi)\}\ ,\\
f_z(Kd,z/d,\phi)&=iKd(z/d)^2(1-z/d)\sin(\phi)\ ,\\
r_1(Kd,\phi)&=1-3\sin^2(\phi)-[\tfrac{1}{6}-\tfrac{7}{30}\sin^2(\phi)](Kd)^2\ ,\\
r_2(Kd,\phi)&=1+3\sin^2(\phi)-[\tfrac{1}{6}+\tfrac{1}{15}\sin^2(\phi)](Kd)^2\ .
\end{aligned} \tag{15}$$

After the substitution of Eqs. (13) and (14) into Eqs. (11) and (12) we arrive at the final result for the frictional force

$$F=F_0\left[1+\frac{1}{Sd^2}\int\frac{d\mathbf{K}}{(2\pi)^2}\{[|\xi_1(\mathbf{K})|^2+|\xi_2(\mathbf{K})|^2]a_1(Kd,\phi)+\xi_1(\mathbf{K})\xi_2(-\mathbf{K})\exp(iK_yV_0t)a_2(Kd,\phi)\}\right]\ , \tag{16}$$

where

$$\begin{aligned}
a_1(Kd,\phi)&=\tfrac{5}{2}-\tfrac{3}{2}\cos(2\phi)-(Kd)^2[\tfrac{7}{10}-\tfrac{1}{30}\cos(2\phi)]+O((Kd)^4)\ ,\\
a_2(Kd,\phi)&=1-3\cos(2\phi)+(Kd)^2[\tfrac{1}{10}+\tfrac{7}{30}\cos(2\phi)]+O((Kd)^4)\ ,
\end{aligned} \tag{17}$$

and $F_0=\eta V_0/d$ is the frictional force for the liquid film between two flat walls [20]. The corrections to the friction due to the roughness of the walls depend on the root mean square heights h_1 and h_2 and the pair correlation functions $g_{ij}(\mathbf{K})$

$$\xi_i(\mathbf{K})\xi_j(-\mathbf{K})=Sh_ih_jg_{ij}(\mathbf{K}),\quad i,j=1,2\ . \tag{18}$$

The functions g_{11} and g_{22} specify the geometrical structures of the bottom and the top walls, respectively, and the cross function g_{12} describes possible correlations between these structures. Equation (16) can be used for the description of the effect both of a random roughness and a periodical corrugation. In the last case we just substituted the integral $\int d\mathbf{K}/(2\pi)^2$ for the sum $S^{-1}\sum_{\mathbf{K}}$ in Eq. (16).

III. DISCUSSION

Equation (16) constitutes the central result in the study of the influence of roughness on the frictional force in boundary lubrication. The frictional force contains two contributions of different origin. The stationary term including the correlation functions $g_{11}(\mathbf{K})$ and $g_{22}(\mathbf{K})$ describes the effect of each rough wall on the hydrodynamical flow in the liquid film. The time-dependent cross term containing $g_{12}(\mathbf{K})$ is due to the interaction between perturbations of the flow at the different walls.

In this section we discuss the dependencies of the frictional force on the correlation properties of roughness. Let us consider the two types of surface morphologies.

A. Periodical corrugation, $\xi_1(\mathbf{R})=h_1\sin(2\pi y/l_1)$ and $\xi_2(\mathbf{R})=h_2\sin(2\pi y/l_2+\gamma)$

For the corrugations with different periods, $l_1\neq l_2$, the frictional force does not depend on the time and has the form

$$F_{st}=\eta\frac{V_0}{d}\left[1+\frac{2}{d^2}(h_1^2+h_2^2)-\frac{22\pi^2}{15}(h_1^2/l_1^2+h_2^2/l_2^2)\right] . \tag{19}$$

For the corrugations with the same periods, $l=l_1=l_2$, there arises the additional time-dependent correction to the frictional force

$$F_t=2\eta h_1h_2\frac{V_0}{d^3}\left[1-\frac{2\pi^2}{15}d^2/l^2\right]\cos(2\pi V_0t/l-\gamma) . \tag{20}$$

Both the time-dependent and the time-independent corrections are of the same order of magnitude and are inverse proportional to the third power of the distance between the walls, d.

The approach developed here can also be applied for studying the effect of microscopic nonuniformity of interface induced by the atomic periodicity of the wall. Atomic scale periodicity of the frictional force has been observed in AFM measurements on the basal plane of graphite [5]. Our consideration demonstrates that the periodical wearless friction can arise in this experiment if water or some other lubricant penetrates into the contact region. The last assumption has already been discussed in Refs. [5,22] and we hope that the future work will clarify this issue.

B. Random roughness

It is often assumed that random roughnesses obey the Gaussian distribution which is characterized by the two parameters: the root mean square height h and the correlation length along the surface l. However, the Gaussian distribution is applicable only for the distances not larger than a few correlation lengths. Considering the relative motion of rough surfaces we have to take into account their properties at the larger scale. At that scale the surface can be represented by the set of identical independent regions, Ω_n, with Gaussian correlations inside each of them. Under this assumption we can write the surface profile functions in the following form:

$$\xi_i(\mathbf{R})=\sum\tilde{\xi}_i(\mathbf{R}-\mathbf{R}_n),\quad |\mathbf{R}_n-\mathbf{R}_m|>l ,$$
$$\langle\tilde{\xi}_i(\mathbf{R}'-\mathbf{R}_n)\tilde{\xi}_j(\mathbf{R}-\mathbf{R}_n)\rangle=h_ih_jg_{ij}(\mathbf{R}-\mathbf{R}') , \tag{21}$$
$$i,j=1,2 .$$

In Eq. (21) the point $\mathbf{R}_n=\{X_n,Y_n\}$ is the center of the region Ω_n, the profile function $\tilde{\xi}_i(\mathbf{R}-\mathbf{R}_n)$ is not equal to zero only for $\mathbf{R}\in\Omega_n$ and the angular brackets denote an ensemble average over various possible configurations in the region Ω_n. The correlation functions g_{ij} are taken in the Gaussian form

$$g_{ij}(\mathbf{K})=\pi l^2\exp(-l^2K^2/4) .$$

Here we assume that both top and bottom surfaces obey the same correlation properties. In this case the frictional force can be written as follows:

$$F=F_{st}+F_t ,$$
$$F_{st}=\eta\frac{V_0}{d}\left[1+\left[\frac{5}{2d^2}-\frac{14}{5l^2}\right](h_1^2+h_2^2)\right] , \tag{22}$$

$$F_t=\eta h_1h_2\frac{V_0}{d^3}\sum_n\left[3l^2/\lambda_n^2+\exp(-\lambda_n^2/l^2)\left[4-3l^2/\lambda_n^2+\frac{2d^2}{15l^2}(4\lambda_n^2/l^2-11)\right]\right] , \tag{23}$$

where $\lambda_n=(Y_n-Y_0-V_0t)$. The time-dependent contribution to the frictional force equals zero in the absence of correlations between the roughnesses of the different walls, but the equation (22) for the stationary contribution conserves its form.

The results obtained demonstrate that the presence of the roughness increases the frictional force and leads to the new dependence of F on the distance between walls. The time-independent correction $F_{st}-F_0$ is proportional to the sum of the mean square heights of roughnesses. We expect that under the experimental conditions, when $d/l\ll 1$, the frictional force depends weakly on the slope of roughness, h/l. For atomically flat solid surfaces the value of h is of the order of 1 nm [18,19]. The surface-force apparatus [2–4] allows the frictional forces to be measured for the average distance between walls is about 5 nm. Under these conditions the time-independent relative correction due to the roughness, $(F_{st}-F_0)/F_0$, equals $\frac{1}{5}$.

The time-dependent contribution to the frictional force arises in the presence of correlations between the geometrical structures of two walls only. This effect is proportional to the product of root mean square heights $h_1 h_2$ and depends both on the short-range (at the scale $\sim l$) and the long-range (at the scale $\sim |\mathbf{R}_n - \mathbf{R}_{n+1}| \gg l$) properties of the roughness. The time dependence of the frictional force is shown in Fig. 2. We see that the presence of the long-range structure leads to the oscillating behavior of the frictional force as a function of time with the extrema at the points $t_n \approx (Y_n - Y_0)/V_0$. For the velocity $V_0 \approx 0.05$ μm/sec and $|Y_n - Y_{n+1}| \approx 1$ μm the interval between the nearest extrema should be about 20 sec. The amplitude of the oscillations is of the order of $F_0/3$ for the values of the parameters used before. We believe that these oscillations can be observed in the SFA experiments.

Unlike our situation, in the experiments [2–4] the bottom wall is coupled through a spring to a stage moving at constant velocity and moves at a variable velocity. However, it should be noted that Eqs. (16), (22), and (23) for the frictional force can also be used for the description of the nonstationary situation if the inequality

$$|dV_0/dt|/V_0 < \eta/\rho d^2 \approx 10^{10}\,\text{sec}^{-1}$$

takes place. It is clear that the last inequality is satisfied in the experiments with the surface force apparatus.

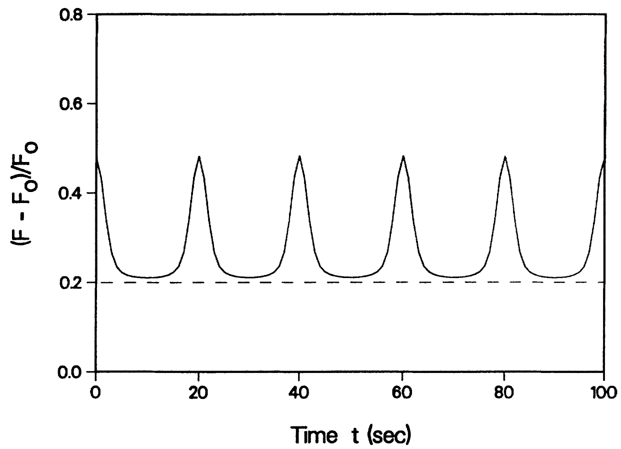

FIG. 2. Time dependence of the frictional force per unit area. The solid line shows the relative correction to the classical result, F_0, due to the roughness; the dashed line is the relative correction frictional force for uncorrelated roughnesses. The calculations were carried out for the following values of the parameters: $h_1 = h_2 = 1$ nm, $d = 5$ nm, $l = 0.1$ μm, $V_0 = 0.05$ μm/sec, $|\mathbf{R}_n - \mathbf{R}_{n+1}| = 1$ μm.

APPENDIX

Using Eqs. (8)–(10) and taking Fourier transforms of both sides of Eqs. (3) we obtain the following.

1. At the movable interface $z = \xi_i(x, y - V_0 t)$

$$\int \frac{d\mathbf{K}}{(2\pi)^2} \Bigg[A_{\alpha,1}(\mathbf{K},\omega')\{\exp[-q'_K \xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}} + A_{\alpha,2}(\mathbf{K},\omega')\{\exp[q'_K \xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}} - \frac{1}{\rho\omega'} K_\alpha (P_1(\mathbf{K},\omega')\{\exp[-K\xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}} + P_2(\mathbf{K},\omega')\{\exp[K\xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}}) \Bigg] = (2\pi) V_0 \delta_{\alpha,y} \delta(\omega - K'_y V_0) \xi_1(\mathbf{K}')/d \;, \quad \text{(A1)}$$

$$\int \frac{d\mathbf{K}}{(2\pi)^2} \Bigg[\frac{i}{q'_K} \sum_\alpha K_\alpha (A_{\alpha,1}(\mathbf{K},\omega')\{\exp[-q'_K \xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}} - A_{\alpha,2}(\mathbf{K},\omega')\{\exp[q'_K \xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}}) - \frac{i}{\rho\omega'} K (P_1(\mathbf{K},\omega')\{\exp[-K\xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}} - P_2(\mathbf{K},\omega')\{\exp[K\xi_1(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}}) \Bigg] = 0 \;, \quad \text{(A2)}$$

where $\omega' = \omega - (K'_y - K_y) V_0$ and $q'_K = q_K(\omega')$.

2. At the fixed interface $z = d + xi_2(x,y)$

$$\int \frac{d\mathbf{K}}{(2\pi)^2} \Bigg[A_{\alpha,1}(\mathbf{K},\omega)(\exp\{-q_K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}} + A_{\alpha,2}(\mathbf{K},\omega)(\exp\{q_K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}} - \frac{1}{\rho\omega} K_\alpha [P_1(\mathbf{K},\omega)(\exp\{-K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}} + P_2(\mathbf{K},\omega)(\exp\{K[d + \xi_2(\mathbf{R}]\})_{\mathbf{K}'-\mathbf{K}}] \Bigg] = (2\pi) V_0 \delta_{\alpha,y} \delta(\omega) \xi_2(\mathbf{K}')/d \;, \quad \text{(A3)}$$

$$\int \frac{d\mathbf{K}}{(2\pi)^2} \Bigg[\frac{i}{q_K} \sum_\alpha K_\alpha [A_{\alpha,1}(\mathbf{K},\omega)(\exp\{-q_K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}} - A_{\alpha,2}(\mathbf{K},\omega)(\exp\{q_K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}}] - \frac{i}{\rho\omega} K [P_1(\mathbf{K},\omega)(\exp\{-K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}} - P_2(\mathbf{K},\omega)(\exp\{K[d + \xi_2(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}}] \Bigg] = 0 \;. \quad \text{(A4)}$$

In Eqs. (A1)–(A4) we introduced the following definition:

$$(\exp\{-p[\xi_j(\mathbf{R})]\})_{\mathbf{K}'-\mathbf{K}}=\int d\mathbf{R}\exp\{-p[\xi_j(\mathbf{R})]\}\exp[-i(\mathbf{K}'-\mathbf{K})\mathbf{R}], \quad j=1,2 \tag{A5}$$

Coupled Eqs. (A1)–(A4) allow us to express coefficients $A_{\alpha,i}(\mathbf{K},\omega)$ and $P_i(\mathbf{K},\omega)$ through the velocity of the bottom wall. We remark that the matrix elements of the type $\{\exp[q_K\xi_j(x,y)]\}_{\mathbf{K}'-\mathbf{K}}$ are not symmetrical functions of $\mathbf{K}$ and $\mathbf{K}'$, since $\mathbf{K}'$ does not appear in the exponent.

We will solve Eqs. (A1)–(A4) within the perturbation theory with respect to the slope of roughness, $|\nabla\xi_j(\mathbf{R})|\ll 1$. At this condition we can expand the matrix elements $(\exp(p\xi_j(x,y))_{\mathbf{K}'-\mathbf{K}}$ in a power series of $p\xi_j(\mathbf{K})$,

$$\{\exp[-p\xi_j(\mathbf{R})]\}_{\mathbf{K}'-\mathbf{K}}=\delta(\mathbf{K}'-\mathbf{K})-p\xi_j(\mathbf{K}'-\mathbf{K})+\tfrac{1}{2}p^2\int\frac{d\mathbf{K}''}{(2\pi)^2}\xi_j(\mathbf{K}'-\mathbf{K}-\mathbf{K}'')\xi_j(\mathbf{K}'')+\cdots , \tag{A6}$$

and solve Eqs. (A1)–(A4) by iterations. As a result we arrive at Eqs. (15) and (16) for the fluid velocity.

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